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COMPUTATION OF QUOTIENT GROUPS OF INVERSE LIMITS OF BURNSIDE RINGS

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Abstract. Let G be a finite group. For a subgroup H of G , we have the Burnside ring $A(H)$ of H . For a set \mathcal{F} of subgroups of G , we have the inverse limit $L(G)$ of the category consisting of modules $A(H)$ ($H \in \mathcal{F}$), and the restriction homomorphism from $A(G)$ to $L(G)$. In this article, we discuss computational theory of the cokernel of the homomorphism.

1. INTRODUCTION

This article is supplementary to the joint work [5] with Masafumi Sugimura.

Let G be a finite group. Let $\mathcal{S}(G)$ denote the set of all subgroups of G and set $\mathcal{F}_G = \mathcal{S}(G) \setminus \{G\}$. Let \mathcal{G} be a subset of $\mathcal{S}(G)$ closed under taking subgroups and conjugations by elements of G . For $H \in \mathcal{S}(G)$, let $A(H)$ denote the Burnside ring of H and $A(H, \mathcal{G} \cap \mathcal{S}(H))$ denote the submodule of $A(H)$ generated by $[H/K]$, where K runs over $\mathcal{G} \cap \mathcal{S}(H)$ (see [2], [4]). We have the inverse limit $L_{\mathcal{F}_G}(G, \mathcal{G}) = \varprojlim_{\mathcal{F}_G} \mathcal{A}$ of the functor $\mathcal{A} : H \mapsto A(H, \mathcal{G} \cap \mathcal{S}(H))$, where $H \in \mathcal{F}_G$, which is a submodule of the cartesian product $P_{\mathcal{F}_G}(G, \mathcal{G}) = \prod_{H \in \mathcal{F}_G} A(H, \mathcal{G} \cap \mathcal{S}(H))$, and the restriction homomorphism

$$\text{res}_{\mathcal{F}_G}^G : A(G, \mathcal{G}) \longrightarrow L_{\mathcal{F}_G}(G, \mathcal{G})$$

(see [3]). Let $Q_{\mathcal{F}_G}(G, \mathcal{G})$ denote the cokernel of $\text{res}_{\mathcal{F}_G}^G$. It is interesting to compute the abelian group $Q_{\mathcal{F}_G}(G, \mathcal{G})$ for a given group G . For a natural number k , let C_k denote a cyclic group of order k . Let p be a prime and let \mathbb{Z}_p denote a module

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consisting of p elements. Let m and n be non-negative integers. The following three computational results have been obtained.

Theorem (Y. Hara–M. Morimoto [3]).

- (1) For $G = C_{p^m}$ ($m \geq 1$), $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p^{m-1}$.
- (2) For $G = C_{p_1 \cdots p_m}$ (p_i distinct primes), $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) = O$.

Theorem (M. Sugimura [6]).

- (1) For $G = C_{p^m} \times C_p$ ($m \geq 1$), $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p^{(m-1)p+1}$.
- (2) For $G = C_{p^m} \times C_{p^2}$ ($m \geq 2$), $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p^{(m-1)(p^2+1)+p+1}$.

Theorem (M. Morimoto–M. Sugimura [5]). For $G = C_{p^m} \times C_{p^n}$ ($m \geq n \geq 2$), $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p^r$, where

$$r = p^n + 2p^{n-1} + \sum_{k=1}^{n-3} (2k+1)p^{n-k-1} + (2n-4)p \\ + (2n-2) + (m-n) \left(\sum_{k=0}^n p^k - p^{n-1} \right).$$

In the present paper, we discuss computational theory to obtain the results above. For example, the next fact follows from Lemma 3.1 and Theorem 6.1.

Theorem 1.1. Let G be a finite (nontrivial) group and N a normal subgroup of G . Suppose each maximal (proper) subgroup H of G contains N . Let \mathcal{K} be the set of all subgroups H of G not containing N . Then it holds that

$$Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong Q_{\mathcal{F}_{\overline{G}}}(\overline{G}, \mathcal{S}(\overline{G})) \oplus Q_{\mathcal{F}_G}(G, \mathcal{K}),$$

where $\overline{G} = G/N$.

The result of Morimoto–Sugimura [5] mentioned above is obtained by using Corollary 7.3.

2. INVERSE LIMIT $L_{\mathcal{F}}(F)$ AND REMARKS

Let $\mathfrak{S}(G)$ denote the subgroup category of G , namely the objects of $\mathfrak{S}(G)$ are all subgroups of G and the morphisms of $\mathfrak{S}(G)$ are all triples (H, g, K) consisting

of $H \in \mathcal{S}(G)$, $g \in G$, $K \in \mathcal{S}(G)$ such that $gHg^{-1} \subset K$. Here (H, g, K) is a morphism from H to K . Each morphism (H, g, K) induces a (group) homomorphism $\iota_{H,g,K} : H \rightarrow K$ by $\iota(a) = gag^{-1}$ for $a \in H$.

Let \mathcal{F} be a subset of $\mathcal{S}(G)$. Let \mathfrak{F} denote the full subcategory of $\mathfrak{S}(G)$ such that $\text{Obj}(\mathfrak{F}) = \mathcal{F}$ and let \mathfrak{Ab} denote the category of abelian groups of which the objects are all abelian groups and the morphisms are all (group) homomorphisms. Let $F : \mathfrak{F} \rightarrow \mathfrak{Ab}$ be a contravariant functor. Then we have the inverse limit $\lim_{\leftarrow \mathcal{F}} F$ of F with respect to \mathcal{F} , where

$$\lim_{\leftarrow \mathcal{F}} F = \left\{ (x_H) \in \prod_{H \in \mathcal{F}} F(H) \mid \begin{array}{l} (H, g, K)^* x_K = x_H \\ \forall (H, g, K) \in \text{Mor}(\mathfrak{F}) \end{array} \right\}$$

(see [1]). This inverse limit will be denoted by $L_{\mathcal{F}}(F)$.

We subsequently use the next fact without specifically mentioning.

Proposition 2.1. *If $F(H)$ is a free \mathbb{Z} -module for each $H \in \mathcal{F}$ then $\lim_{\leftarrow \mathcal{F}} F$ is a direct summand of $\prod_{H \in \mathcal{F}} F(H)$.*

Let e denote the identity element of G . For a morphism (H, e, K) of $\mathfrak{S}(G)$, let res_H^K denote the homomorphism $(H, e, G)^* : F(K) \rightarrow F(H)$. Let $P_{\mathcal{F}}(F)$ denote the cartesian product $\prod_{H \in \mathcal{F}} F(H)$. The image of the restriction homomorphism $\text{res}^G = \prod_{H \in \mathcal{F}} \text{res}_H^G : F(G) \rightarrow P_{\mathcal{F}}(F)$ is contained in $L_{\mathcal{F}}(F)$ and therefore res^G induces $\text{res}_{\mathcal{F}}^G : F(G) \rightarrow L_{\mathcal{F}}(F)$. Let $B_{\mathcal{F}}(F)$ denote the image of $\text{res}_{\mathcal{F}}^G$ and $Q_{\mathcal{F}}(F)$ the quotient group $L_{\mathcal{F}}(F)/B_{\mathcal{F}}(F)$, namely the cokernel of $\text{res}_{\mathcal{F}}^G$.

Let \mathcal{F}^* denote a complete set of representatives of G -conjugacy classes of subgroups belonging to \mathcal{F} such that $\mathcal{F}^* \subset \mathcal{F}$. Let $\text{proj} : \prod_{H \in \mathcal{F}} F(H) \rightarrow \prod_{H \in \mathcal{F}^*} F(H)$ be the canonical projection. This gives the homomorphisms $\text{proj}_L : L_{\mathcal{F}}(F) \rightarrow L_{\mathcal{F}^*}(F)$ and $\text{proj}_B : B_{\mathcal{F}}(F) \rightarrow B_{\mathcal{F}^*}(F)$ as well as the homomorphism $\rho_Q : Q_{\mathcal{F}}(F) \rightarrow Q_{\mathcal{F}^*}(F)$. Clearly, the diagram

$$\begin{array}{ccccc} B_{\mathcal{F}}(F) & \hookrightarrow & L_{\mathcal{F}}(F) & \twoheadrightarrow & Q_{\mathcal{F}}(F) \\ \text{proj}_B \downarrow & & \downarrow \text{proj}_L & & \downarrow \rho_Q \\ B_{\mathcal{F}^*}(F) & \hookrightarrow & L_{\mathcal{F}^*}(F) & \twoheadrightarrow & Q_{\mathcal{F}^*}(F) \end{array}$$

commutes.

Proposition 2.2. *In the diagram above, proj_B , proj_L , and ρ_Q all are isomorphisms.*

Proof. By definition of inverse limit, proj_L is an isomorphism. Since proj_B is surjective and proj_L is injective, proj_B is bijective. As proj_L and proj_B are isomorphisms, ρ_Q is an isomorphism. \square

3. DEFINITION OF $Q_{\mathcal{F}}(G, \mathcal{G})$ AND INTERPRETATION OF $L_{\mathcal{F}}(G, \mathcal{G})$

Let \mathcal{G} be a subset of $\mathcal{S}(G)$ closed under taking conjugations by elements in G . For a subgroup H of G , let $\mathcal{G}(\cap H)$ denote the set consisting of all $K \cap H$, where K ranges over \mathcal{G} . We have the restriction homomorphism $\text{res}_H^G : A(G, \mathcal{G}) \rightarrow A(H, \mathcal{G}(\cap H))$. Let $P_{\mathcal{F}}(G, \mathcal{G})$ denote the cartesian product $\prod_{H \in \mathcal{F}} A(H, \mathcal{G}(\cap H))$. We have the restriction homomorphism $\text{res}^G : A(G, \mathcal{G}) \rightarrow P_{\mathcal{F}}(G, \mathcal{G})$. Let $L_{\mathcal{F}}(G, \mathcal{G})$ denote the inverse limit $\varprojlim_{\mathcal{F}} \mathcal{A}$ of the Burnside ring functor $\mathcal{A} : H \mapsto A(H, \mathcal{G}(\cap H))$, where $H \in \mathcal{S}(G)$. Since the image of res^G above is contained in $L_{\mathcal{F}}(G, \mathcal{G})$, we obtain the restriction homomorphism $\text{res}_{\mathcal{F}}^G : A(G, \mathcal{G}) \rightarrow L_{\mathcal{F}}(G, \mathcal{G})$. Let $B_{\mathcal{F}}(G, \mathcal{G})$ denote the image of $\text{res}_{\mathcal{F}}^G$ and let $Q_{\mathcal{F}}(G, \mathcal{G})$ denote the cokernel of $\text{res}_{\mathcal{F}}^G$, i.e. $Q_{\mathcal{F}}(G, \mathcal{G}) = L_{\mathcal{F}}(G, \mathcal{G}) / B_{\mathcal{F}}(G, \mathcal{G})$.

Lemma 3.1. *Let \mathcal{F} be a subset of $\mathcal{S}(G)$ closed under taking subgroups, and let \mathcal{G} be a subset of $\mathcal{S}(G)$ closed under taking subgroups and conjugations by elements of G . Then $L_{\mathcal{F}}(G, \mathcal{G})$ coincides with $\overline{B_{\mathcal{F}}(G, \mathcal{G})}$, where*

$$\overline{B_{\mathcal{F}}(G, \mathcal{G})} = \{x \in P_{\mathcal{F}}(G, \mathcal{G}) \mid nx \in B_{\mathcal{F}}(G, \mathcal{G}) \text{ for some } n \in \mathbb{N}\}.$$

Proof. First recall that $L_{\mathcal{F}}(G, \mathcal{G})$ is a direct summand of the \mathbb{Z} -free module $P_{\mathcal{F}}(G, \mathcal{G})$. The lemma above follows from the facts that $B_{\mathcal{F}}(G, \mathcal{G}) \subset L_{\mathcal{F}}(G, \mathcal{G})$ and that $\text{rank}_{\mathbb{Z}} B_{\mathcal{F}}(G, \mathcal{G}) = \text{rank}_{\mathbb{Z}} L_{\mathcal{F}}(G, \mathcal{G})$. \square

4. DEFINITION OF GROUPS $R(-)$ AND REMARKS

Let W be a finitely generated free \mathbb{Z} -module. For a submodule U of W , we define the submodule \overline{U} of W as $\{x \in W \mid nx \in U \text{ for some } n \in \mathbb{N}\}$. Therefore \overline{U} is the smallest direct summand of W containing U . We define a finite module $R_W(U)$ by

$$(4.1) \quad R_W(U) = \overline{U}/U.$$

Clearly, $R_W(U)$ coincides with $R_{\overline{U}}(U)$. We readily see

Proposition 4.1. *Let $U = U_1 \oplus U_2$ be a submodule of W .*

- (1) \overline{U} contains $\overline{U_1} + \overline{U_2}$.
- (2) If $\overline{U} = \overline{U_1} + \overline{U_2}$ holds then $R_W(U) = R_W(U_1) \oplus R_W(U_2)$.

Let $f : V \rightarrow W$ be a homomorphism between finitely generated free \mathbb{Z} -modules V and W . We define a finite module $R(f)$ by

$$(4.2) \quad R(f) = R_W(f(V)) \quad (= \overline{f(V)}/f(V)).$$

We immediately obtain

Proposition 4.2. *Let V, W, V' , and W' be finitely generated free \mathbb{Z} -modules, and let $f : V \rightarrow W$ be a homomorphism.*

- (1) If $\alpha : V' \rightarrow V$ is an epimorphism then $R(f) = R(f \circ \alpha)$.
- (2) If $\beta : W \rightarrow W'$ is a homomorphism such that $\beta|_{\overline{f(V)}} : \overline{f(V)} \rightarrow W'$ is split injective then $R(f) \cong R(\beta \circ f)$.

Let $M = [u_{ij}]$ be an $m \times n$ -matrix with entries in \mathbb{Z} , i.e. $M \in M_{m,n}(\mathbb{Z})$. Then we have the homomorphism $f_M : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ defined by $f_M(\mathbf{x}) = \mathbf{x}M$, where $\mathbf{x} = [x_1, \dots, x_m] \in \mathbb{Z}^m$. We define the finite module $R(M)$ by

$$(4.3) \quad R(M) = R(f_M).$$

Therefore $R(M)$ coincides with $R_{\mathbb{Z}^n}(U)$, where $U = \langle \mathbf{u}_1, \dots, \mathbf{u}_m \rangle_{\mathbb{Z}}$ with $\mathbf{u}_i = [u_{i1}, \dots, u_{in}]$ ($i = 1, \dots, m$).

Example 4.3. For natural numbers p_1, \dots, p_n and the matrix

$$M = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & p_n & 0 & \cdots & 0 \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

$R(M)$ is isomorphic to $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$.

We remark that in general, the module $R\left(\begin{bmatrix} M_{11} & M_{12} \\ O & M_{22} \end{bmatrix}\right)$ is not isomorphic to $R(M_{11}) \oplus R(M_{22})$, nor to $R\left(\begin{bmatrix} M_{11} & M_{12} \end{bmatrix}\right) \oplus R(M_{22})$.

Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ and $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be \mathbb{Z} -bases of V and W , respectively. For a homomorphism $f : V \rightarrow W$, the matrix $M_f = [u_{ij}] \in M_{m,n}(\mathbb{Z})$ is defined by

$$f(\mathbf{a}_i) = \sum_{j=1}^n u_{ij} \mathbf{b}_j \quad (i = 1, \dots, m).$$

The matrix M_f is called the *matrix presentation* of f with respect to \mathcal{A} and \mathcal{B} . We readily obtain

Proposition 4.4. *Let $f : V \rightarrow W$ be a homomorphism and let $M_f = [u_{ij}] \in M_{m,n}(\mathbb{Z})$ be the matrix presentation of f . Then the module $R(f)$ is isomorphic to $R(M_f)$, and hence to $R_{\mathbb{Z}^n}(U)$, where $U = \langle \mathbf{u}_1, \dots, \mathbf{u}_n \rangle_{\mathbb{Z}}$ and $\mathbf{u}_i = [u_{i1}, \dots, u_{in}]$ ($i = 1, \dots, n$).*

For matrices $M, N \in M_{m,n}(\mathbb{Z})$, we say that M is *similar* to N , and write $M \sim N$, if there exist $X \in GL_m(\mathbb{Z})$ and $Y \in GL_n(\mathbb{Z})$ such that $N = X \cdot M \cdot Y$. By Proposition 4.2, we get

Proposition 4.5. *Let M and N be matrices in $M_{m,n}(\mathbb{Z})$. If M is similar to N then $R(M)$ is isomorphic to $R(N)$.*

Corollary 4.6. *Let $X \in M_{p,q}(\mathbb{Z})$, $Y \in M_{s,t}(\mathbb{Z})$, and $Z \in M_{q,t}(\mathbb{Z})$. Then*

$$R\left(\begin{bmatrix} X & X \cdot Z \\ O & Y \end{bmatrix}\right) \cong R\left(\begin{bmatrix} X & O \\ O & Y \end{bmatrix}\right) \cong R(X) \oplus R(Y).$$

Proof. First note

$$\begin{bmatrix} I & Z \\ O & I \end{bmatrix} \in GL_{q+t}(\mathbb{Z}).$$

The corollary above follows from the equality

$$\begin{bmatrix} X & X \cdot Z \\ O & Y \end{bmatrix} = \begin{bmatrix} X & O \\ O & Y \end{bmatrix} \begin{bmatrix} I & Z \\ O & I \end{bmatrix}$$

and Proposition 4.5. □

We give a computational example of Proposition 4.5.

Example 4.7. Let p be a natural number and M the $(p+3) \times (2p+2)$ -matrix

$$\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\ p & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & p & 0 & & 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & p & 0 & \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & p & 1 & \cdots & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & p & \cdots & \cdots & p & p \end{bmatrix}.$$

We readily check that M is similar to the matrix

$$N = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & p & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Therefore we get $R(M) \cong R(N) \cong \mathbb{Z}_p$.

5. COINCIDENCE OF $R(\text{res}_{\mathcal{F}}^G)$ AND $R(\text{res}_{\mathcal{F}_{\max}^*}^G)$

For $\mathcal{F} \subset \mathcal{S}(G)$, let \mathcal{F}_{\max} be the set of all maximal elements of \mathcal{F} and let \mathcal{F}^* be a complete set of representatives of conjugacy classes of subgroups belonging to \mathcal{F} such that $\mathcal{F}^* \subset \mathcal{F}$. If \mathcal{F}_{\max} is closed under taking conjugations by elements of G , the equality $\mathcal{F}_{\max} = \bigcup_{H \in \mathcal{F}_{\max}^*} (H)$ holds, where $(H) = \{gHg^{-1} \mid g \in G\}$. Let \mathcal{G} be a subset of $\mathcal{S}(G)$ closed under taking conjugations by elements in G . We have the commutative diagram

$$\begin{array}{ccccc} A(G, \mathcal{G}) & \xrightarrow{\text{res}_{\mathcal{F}}^G} & L_{\mathcal{F}}(G, \mathcal{G}) & \xrightarrow{j_1} & P_{\mathcal{F}}(G, \mathcal{G}) \\ & \searrow \text{res}_{\mathcal{F}_{\max}^*}^G & \downarrow \eta & & \downarrow \text{proj} \\ & & L_{\mathcal{F}_{\max}^*}(G, \mathcal{G}) & \xrightarrow{j_2} & P_{\mathcal{F}_{\max}^*}(G, \mathcal{G}) \end{array}$$

consisting of canonical homomorphisms.

Proposition 5.1. *For the diagram above, the following holds.*

- (1) *The homomorphisms j_1 and j_2 are split injective.*
- (2) *The composition $\text{proj} \circ j_1 : L_{\mathcal{F}}(G, \mathcal{G}) \rightarrow P_{\mathcal{F}_{\max}^*}(G, \mathcal{G})$ is split injective.*

$$(3) \ R(\text{res}_{\mathcal{F}}^G) \cong R(\text{res}_{\mathcal{F}_{\max}}^G) \cong R(j_2 \circ \text{res}_{\mathcal{F}_{\max}}^G).$$

Proof. Since $A(H)$ is a free \mathbb{Z} -module for each $H \in \mathcal{F}_{\max}$, j_1 and j_2 are split injective, which implies $R(\text{res}_{\mathcal{F}_{\max}}^G) \cong R(j_2 \circ \text{res}_{\mathcal{F}_{\max}}^G)$. Since j_2 and η both are split injective, $\text{proj} \circ j_1$ is split injective and hence $R(\text{res}_{\mathcal{F}}^G) \cong R(\text{proj} \circ j_1 \circ \text{res}_{\mathcal{F}}^G)$. Thus we conclude $R(\text{res}_{\mathcal{F}}^G) \cong R(\text{res}_{\mathcal{F}_{\max}}^G)$. \square

The next corollary immediately follows from the proposition above.

Corollary 5.2. *For the commutative diagram*

$$\begin{array}{ccccc}
 & & L_{\mathcal{F}}(G, \mathcal{G}) & & \\
 & \nearrow \text{res}_{\mathcal{F}}^G & \downarrow \eta_1 & & \\
 A(G, \mathcal{G}) & \xrightarrow{\text{res}_{\mathcal{F}_{\max}}^G} & L_{\mathcal{F}_{\max}}(G, \mathcal{G}) & & \\
 & \searrow \text{res}_{\mathcal{F}_{\max}}^G & \downarrow \eta_2 & & \\
 & & L_{\mathcal{F}_{\max}}^*(G, \mathcal{G}) & \xrightarrow{j} & P_{\mathcal{F}_{\max}}^*(G, \mathcal{G})
 \end{array}$$

consisting of canonical homomorphisms, it holds that

$$R(\text{res}_{\mathcal{F}}^G) \cong R(\text{res}_{\mathcal{F}_{\max}}^G) \cong R(\text{res}_{\mathcal{F}_{\max}}^G \circ j) \cong R(j \circ \text{res}_{\mathcal{F}_{\max}}^G).$$

Example 5.3 (cf. [3, Proposition 2.2]). Let p be a prime, C_p a cyclic group of order p , $G = C_p \times C_p$, $\mathcal{F} = \mathcal{F}_G$, and M the matrix given in Example 4.7. Then it holds that

$$(5.1) \quad Q_{\mathcal{F}}(G, \mathcal{S}(G)) \cong R(\text{res}_{\mathcal{F}}^G) \cong R(j \circ \text{res}_{\mathcal{F}_{\max}}^G) \cong R(M) \cong \mathbb{Z}_p.$$

6. DECOMPOSITION OF $R(\text{res}_{\mathcal{F}}^G)$

In this section, let N be a normal subgroup of G and let \mathcal{F} and \mathcal{G} be subsets of $\mathcal{S}(G)$ closed under taking conjugations by elements of G . We use the notation

$$\begin{aligned}
 \mathcal{F}(\geq N) &= \{H \in \mathcal{F} \mid H \supset N\}, \\
 \mathcal{F}(\geq N)/N &= \{H/N \mid H \in \mathcal{F}(\geq N)\}, \\
 \mathcal{F}(\geq N)' &= \mathcal{F} \setminus \mathcal{F}(\geq N).
 \end{aligned}$$

Let

$$\begin{aligned} \text{res}_{\mathcal{F}}^G &: A(G, \mathcal{G}) \rightarrow L_{\mathcal{F}}(G, \mathcal{G}), \\ \text{res}_1 &: A(G, \mathcal{G}(\geq N)) \rightarrow L_{\mathcal{F}(\geq N)}(G, \mathcal{G}(\geq N)), \\ \text{res}_2 &: A(G, \mathcal{G}(\geq N)') \rightarrow L_{\mathcal{F}}(G, \mathcal{G}(\geq N)'), \text{ and} \\ \overline{\text{res}}_1 &: A(G/N, \mathcal{G}(\geq N)/N) \rightarrow L_{\mathcal{F}(\geq N)/N}(G/N, \mathcal{G}(\geq N)/N) \end{aligned}$$

denote the restriction homomorphisms, respectively.

Theorem 6.1. *If the condition $N \subset \bigcap_{L \in \mathcal{F}_{\max}} L$ is satisfied then it holds that*

$$R(\text{res}_{\mathcal{F}}^G) \cong R(\text{res}_1) \oplus R(\text{res}_2) \cong R(\overline{\text{res}}_1) \oplus R(\text{res}_2).$$

Proof. We can readily see $R(\text{res}_1) \cong R(\overline{\text{res}}_1)$. Observe the commutative diagram

$$(6.1) \quad \begin{array}{ccccc} A(G, \mathcal{G}(\geq N)) & \xrightarrow{\text{res}_1} & L_{\mathcal{F}(\geq N)}(G, \mathcal{G}(\geq N)) & \xrightarrow{\rho_1} & P_{\mathcal{F}_{\max}}^*(G, \mathcal{G}(\geq N)) \\ j_1 \downarrow & & \downarrow \eta_1 & & \downarrow \iota_1 \\ A(G, \mathcal{G}) & \xrightarrow{\text{res}_{\mathcal{F}}^G} & L_{\mathcal{F}}(G, \mathcal{G}) & \xrightarrow{\rho} & P_{\mathcal{F}_{\max}}^*(G, \mathcal{G}) \\ j_2 \uparrow & & \uparrow \eta_2 & & \uparrow \iota_2 \\ A(G, \mathcal{G}(\geq N)') & \xrightarrow{\text{res}_2} & L_{\mathcal{F}}(G, \mathcal{G}(\geq N)') & \xrightarrow{\rho_2} & P_{\mathcal{F}_{\max}}^*(G, \mathcal{G}(\geq N)') \end{array}$$

consisting of canonical homomorphisms. By Proposition 5.1, we have $R(\text{res}_{\mathcal{F}}^G) \cong R(\rho \circ \text{res}_{\mathcal{F}}^G)$, $R(\text{res}_1) \cong R(\rho_1 \circ \text{res}_1)$, and $R(\text{res}_2) \cong R(\rho_2 \circ \text{res}_2)$. There are canonical direct sum decompositions

$$\begin{aligned} A(G, \mathcal{G}) &= A(G, \mathcal{G}(\geq N)) \oplus A(G, \mathcal{G}(\geq N)'), \text{ and} \\ P_{\mathcal{F}_{\max}}^*(G, \mathcal{G}) &= P_{\mathcal{F}_{\max}}^*(G, \mathcal{G}(\geq N)) \oplus P_{\mathcal{F}_{\max}}^*(G, \mathcal{G}(\geq N)'). \end{aligned}$$

With respect to these direct sums, $\rho \circ \text{res}_{\mathcal{F}}^G$ coincides with $(\rho_1 \circ \text{res}_1) \oplus (\rho_2 \circ \text{res}_2)$. Thus we get $R(\rho \circ \text{res}_{\mathcal{F}}^G) \cong R(\rho_1 \circ \text{res}_1) \oplus R(\rho_2 \circ \text{res}_2)$. \square

Example 6.2. Let p be a prime and m a natural number ≥ 2 . Let a and b be generators of C_{p^m} and C_p , respectively, and let $G = C_{p^m} \times C_p$ be the group generated by a and b . Let N be the subgroup of G generated by $a^{p^{m-1}}$. We regard $N = C_p \times E$ as the subgroup of $C_{p^m} \times C_p$. Let $\mathcal{F} = \mathcal{F}_G$, $K = G/N$, and $\mathcal{H} = \mathcal{F}_K$. We remark

that $K \cong C_{p^{m-1}} \times C_p$. Let M be the $(p+1) \times (2p+1)$ -matrix

$$\begin{bmatrix} p & 0 & \cdots & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & p & \ddots & & \vdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & p & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & p & p & p & \cdots & p \end{bmatrix}.$$

We readily show that M is similar to the matrix

$$\begin{bmatrix} p & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p & \ddots & & \vdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & p & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus we get $R(M) \cong \mathbb{Z}_p^p$. It follows that

$$\begin{aligned} (6.2) \quad Q_{\mathcal{F}}(G, \mathcal{S}(G)) &\cong R(\text{res}_{\mathcal{F}}^G) \cong R(\text{res}_{\mathcal{F}_{\max}^*}^G) \cong R(\text{res}_{\mathcal{H}_{\max}^*}^K) \oplus R(M) \\ &\cong Q_{\mathcal{H}}(K, \mathcal{S}(K)) \oplus \mathbb{Z}_p^p. \end{aligned}$$

7. DECOMPOSITION OF $R(\text{res}_{\mathcal{F}}^G)$ FOR $G = C_{p^m} \times C_{p^n}$

Throughout this section, let $G = C_{p^m} \times C_{p^n}$ with $m \geq n \geq 2$ for a prime p . Let a and b be generators of the cyclic groups C_{p^m} and C_{p^n} , respectively. Let N denote the subgroup generated by $a^{p^{m-1}}$ and $b^{p^{n-1}}$. Thus N is isomorphic to $C_p \times C_p$. Let H_0 denote the subgroup generated by $a^{p^{m-1}}$. We can regard H_0 as the subgroup $C_p \times E$ of $C_{p^m} \times C_{p^n}$, where E is the trivial group. The group G also contains subgroups H_i ($i = 1, \dots, p$) of order p generated by $a^{ip^{m-1}}b^{p^{n-1}}$. We can regard H_p as the subgroup $E \times C_p$ of $C_{p^m} \times C_{p^n}$. Let

$$\mathcal{G} = \mathcal{S}(G),$$

$$\mathcal{G}_1 = \{H \in \mathcal{S}(G) \mid H \supset N\},$$

$$\mathcal{G}_{2,i} = \{H \in \mathcal{S}(G) \setminus \mathcal{G}_1 \mid H \supset H_i\},$$

where $i = 0, \dots, p$. Then the equality

$$(7.1) \quad \mathcal{G} = \mathcal{G}_1 \amalg \mathcal{G}_{2,0} \amalg \left(\prod_{i=1}^p \mathcal{G}_{2,i} \right) \amalg \{E\}$$

holds, which gives the decomposition formula

$$(7.2) \quad A(G, \mathcal{G}) = A(G, \mathcal{G}_1) \oplus A(G, \mathcal{G}_{2,0}) \oplus \bigoplus_{i=1}^p A(G, \mathcal{G}_{2,i}) \oplus A(G, \{E\})$$

of the Burnside module. In addition, we have the canonical identifications

$$(7.3) \quad \begin{aligned} \mathcal{G}_1 &= \mathcal{S}(G/N), \\ \mathcal{G}_1 \cup \mathcal{G}_{2,0} &= \mathcal{S}(G/H_0), \\ G/N &= C_{p^{m-1}} \times C_{p^{n-1}}, \text{ and} \\ G/H_0 &= C_{p^{m-1}} \times C_{p^n}. \end{aligned}$$

Set $X_{m,n} = A(G, \mathcal{G}_1)$, $Y_{m,n,i} = A(G, \mathcal{G}_{2,i})$ ($i = 0, \dots, p$), and $Z_{m,n} = A(G, \{E\})$. By (7.2) we get

$$(7.4) \quad A(C_{p^m} \times C_{p^n}) = X_{m,n} \oplus \bigoplus_{i=0}^p Y_{m,n,i} \oplus Z_{m,n}.$$

It follows from (7.3) that

$$(7.5) \quad \begin{aligned} A(C_{p^{m-1}} \times C_{p^{n-1}}) &\cong X_{m,n}, \\ A(C_{p^{m-1}} \times C_{p^n}) &\cong X_{m,n} \oplus Y_{m,n,0} \end{aligned}$$

In addition, we have

$$(7.6) \quad Y_{m,n,i} \cong Y_{n,n,i} \cong Y_{n,n,0} \quad (i = 1, \dots, p).$$

Let $\mathcal{F} = \mathcal{S}(G) \setminus \{G\}$, $\mathcal{F}_1 = \{H \in \mathcal{F} \mid H \supset N\}$, $\mathcal{F}_{2,i} = \{H \in \mathcal{F} \mid H \supset H_i\}$ ($i = 0, \dots, p$), and let $f_{m,n}$, $g_{m,n}$, $h_{m,n,i}$, $k_{m,n}$ be the restriction homomorphisms

$$\begin{aligned} \text{res}_{\mathcal{F}}^G &: A(G) \rightarrow L_{\mathcal{F}}(G, \mathcal{S}(G)), \\ \text{res}_{\mathcal{F}_1}^G &: X_{m,n} \rightarrow L_{\mathcal{F}_1}(G, \mathcal{G}_1), \\ \text{res}_{\mathcal{F}_{2,i}}^G &: Y_{m,n,i} \rightarrow L_{\mathcal{F}_{2,i}}(G, \mathcal{G}_{2,i}), \\ \text{res}_{\mathcal{F}}^G &: Z_{m,n} \rightarrow L_{\mathcal{F}}(G, \{E\}), \end{aligned}$$

respectively.

Theorem 7.1. *Under the situation above, the direct sum decomposition formula*

$$R(f_{m,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0}) \oplus \bigoplus_{i=1}^p R(h_{m,n,i}) \oplus R(k_{m,n})$$

holds.

Proof. Similarly to the diagram (6.1), we have diagrams

$$\begin{array}{ccccc}
 X_{m,n} & \xrightarrow{g_{m,n}} & L_{\mathcal{F}_1}(G, \mathcal{G}_1) & \longrightarrow & P_{\mathcal{F}_{\max}}(G, \mathcal{G}_1) \\
 \downarrow & & \downarrow & & \downarrow \\
 A(G) & \xrightarrow{f_{m,n}} & L_{\mathcal{F}}(G, \mathcal{G}) & \longrightarrow & P_{\mathcal{F}_{\max}}(G, \mathcal{G}) \\
 \uparrow & & \uparrow & & \uparrow \\
 A(G, \mathcal{G}_1') & \xrightarrow{g_{m,n}'} & L_{\mathcal{F}}(G, \mathcal{G}_1') & \longrightarrow & P_{\mathcal{F}_{\max}}(G, \mathcal{G}_1'), \\
 \\
 Y_{m,n,0} & \xrightarrow{h_{m,n,0}} & L_{\mathcal{F}_{2,0}}(G, \mathcal{G}_{2,0}) & \longrightarrow & P_{\mathcal{F}_{\max}}(G, \mathcal{G}_{2,0}) \\
 \downarrow & & \downarrow & & \downarrow \\
 A(G, \mathcal{G}_1') & \xrightarrow{g_{m,n}'} & L_{\mathcal{F}}(G, \mathcal{G}_1') & \longrightarrow & P_{\mathcal{F}_{\max}}(G, \mathcal{G}_1') \\
 \uparrow & & \uparrow & & \uparrow \\
 A(G, \mathcal{G}_{2,0}') & \xrightarrow{h_{m,n,0}'} & L_{\mathcal{F}}(G, \mathcal{G}_{2,0}') & \longrightarrow & P_{\mathcal{F}_{\max}}(G, \mathcal{G}_{2,0}'),
 \end{array}$$

and so on, where $\mathcal{G}_1' = \mathcal{G} \setminus \mathcal{G}_1$ and $\mathcal{G}_{2,0}' = \mathcal{G}_1' \setminus \mathcal{G}_{2,0}$. The theorem above is obtained by iteration of use of Theorem 6.1. \square

Concerning with Theorem 7.1, we remark

Proposition 7.2.

- (1) $R(f_{m-1,n-1}) \cong R(g_{m,n})$.
- (2) $R(f_{m-1,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0})$.
- (3) $R(f_{m,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0}) \oplus R(h_{n,n,0})^{\oplus p} \oplus \mathbb{Z}_p$.

Proof. The claims (1) and (2) follow from (7.5). Since $R(k_{m,n}) \cong \mathbb{Z}_p$, the claim (3) follows from Theorem 7.1 and (7.6). \square

The next fact immediately follows from the proposition above.

Corollary 7.3.

- (1) $R(f_{n-1,n}) \cong R(f_{n-1,n-1}) \oplus R(h_{n,n,0})$.
- (2) $R(f_{m,n}) \cong R(f_{m-1,n}) \oplus R(h_{n,n,0})^{\oplus p} \oplus \mathbb{Z}_p$.

Now recall Lemma 3.1 and Sugimura's theorem described in the introduction. By induction arguments on m and n with the corollary above, we can readily prove

Proposition 7.4. *Any element $x \in Q_{\mathcal{F}_G}(G, \mathcal{S}(G))$ has exponent p , i.e. $px = 0$, and hence $Q_{\mathcal{F}_G}(G, \mathcal{S}(G))$ is isomorphic to a direct sum of copies of \mathbb{Z}_p .*

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